Problem set 2
Due date: 28th Jan

Part A (submit any three)

Exercise 10. (1) Let $X:\mathbb{R}^n\to\mathbb{R}^m$ be a continuous function. Show that X is a random variable. Show that X is a r.v. if it is (a) right continuous or (b) lower semicontinuous or (c) non-decreasing (take $m=n=1$ for the last one).
(2) If μ is a Borel p.m. on $\mathbb{R}$ with CDF F, then find the push-forward of μ under F.

Exercise 11. Show that composition of random variables is a random variable. Show that real-valued random variables on a given $(\Omega,\mathcal{F})$ are closed under linear combinations, under multiplication, under countable suprema (or infima) and under limsup (or liminf) of countable sequences.

Exercise 12. Let $\mu_n=\frac{1}{n}\sum_{k=1}^{n}\delta_{k/n}$ and let μ be the uniform p.m. Show directly by definition that $d(\mu_n,\mu)\to 0$ as $n\to\infty$.

Exercise 13 (Change of variable for densities). (1) Let μ be a p.m. on $\mathbb{R}$ with density f by which we mean that its CDF $F_\mu(x)=\int_{-\infty}^{x}f(t)dt$ (you may assume that f is continuous, non-negative and the Riemann integral $\int_{\mathbb{R}}f=1$). Then, find the (density of the) push forward measure of μ under (a) $T(x)=x+a$ (b) $T(x)=bx$ (c) T is any increasing and differentiable function.
(2) If X has $N(\mu,\sigma^2)$ distribution, find the distribution of $(X-\mu)/\sigma$.

Exercise 14. (1) Let $X=(X_1,\dots,X_n)$. Show that X is an $\mathbb{R}^d$-valued r.v. if and only if $X_1,\dots,X_n$ are (real-valued) random variables. How does $\sigma(X)$ relate to $\sigma(X_1),\dots,\sigma(X_n)$?
(2) Let $X:\Omega_1\to\Omega_2$ be a random variable. If $X(\omega)=X(\omega')$ for some $\omega,\omega'\in\Omega_1$, show that there is no set $A\in\sigma(X)$ such that $\omega\in A$ and $\omega'\notin A$ or vice versa. [**Extra!** If $Y:\Omega_1\to\Omega_2$ is another r.v. which is measurable w.r.t. $\sigma(X)$ on Ω_1, then show that Y is a function of X].

Part B (submit any two)

Exercise 15 (Levy metric). (1) Show that the Lévy metric on $\mathcal{P}(\mathbb{R}^d)$ defined in class is actually a metric.
(2) Show that under the Lévy metric, $\mathcal{P}(\mathbb{R}^d)$ is a complete and seperable metric space.

Exercise 16. On the probabiity space $([0,1],\mathcal{B},\mathbf{m})$, for $k\geq 1$, define the functions

$$X_k(t):=\begin{cases}0 & \text{if } t\in\bigcup_{j=0}^{2^{k-1}-1}[\frac{2j}{2^k},\frac{2j+1}{2^k}).\\ 1 & \text{if } t\in\bigcup_{j=0}^{2^{k-1}-1}[\frac{2j+1}{2^k},\frac{2j+2}{2^k}) \text{ or } t=1.\end{cases}$$

(1) For any $n\geq 1$, what is the distribution of X_n?
(2) For any fixed $n\geq 1$, find the joint distribution of $(X_1,\dots,X_n)$.

[**Note:** $X_k(t)$ is just the k^{th} digit in the binary expansion of t. Dyadic rationals have two binary expansions, and we have chosen the finite expansion (except at $t=1$)].

Exercise 17 (Coin tossing space). Continuing with the previous example, consider the mapping $X:[0,1]\to\{0,1\}^{\mathbb{N}}$ defined by $X(t)=(X_1(t),X_2(t),\dots)$. With the Borel σ-algebra on $[0,1]$ and the σ-algebra generated by cylinder sets on $\{0,1\}^{\mathbb{N}}$, show that X is a random variable and find the push-foward of the Lebesue measure under X.

Exercise 18 (Equivalent conditions for weak convergence). Show that the following statements are equivalent to $\mu_n\xrightarrow{d}\mu$.
(1) $\limsup_{n\to\infty}\mu_n(F)\leq\mu(F)$ if F is closed.
(2) $\liminf_{n\to\infty}\mu_n(G)\geq\mu(G)$ if G is open.
(3) $\limsup_{n\to\infty}\mu_n(A)=\mu(A)$ if $A\in\mathcal{F}$ and $\mu(\partial A)=0$.